Road to Quantum Field Theory

November 15, 2007

Quantum field theory is a combination of two most successful theories a.k.a. quantum mechanics and special relativity. This is not a trivial fact, if we give a further thought why Einstein's another adult child a.k.a. general relativistic theory of gravity can not be simply included.

1 Quantum Mechanics

Let us start from the review of quantum mechanics.

Foundation or general recipe for a quantum theory is:

1. A Hilbert space of physical states $|\psi>$, $\mathcal{H}$, whose inner product is defined as $<\psi|\chi>$.

2. Principle of superposition, if $|\psi>$ and $|\chi>$ are physical states, so is $\alpha|\psi>+\beta|\chi>$

3. Observables are Hermition Operators $\mathcal{O}$ on Hilbert space $\mathcal{H}$, here Hermition condition can be expressed as:

$$<\psi|O|\chi>=<\chi|O|\psi>^*$$

Every possible values of observables are $\mathcal{O}$'s eigenvalue, which is real.

4. States in Hilbert space can always be decomposed into series of completed basis $|n>$, whose eigenvalues are λ_n respectively.

$$|\psi>=\sum \alpha_n|n>$$

The probability of observing λ_n is

$$P=\frac{|\alpha_n|^2}{\sum\limits_m |\alpha_m|^2}$$

where

$$\alpha_n =< n|\psi > \qquad \text{and} \quad < n|n >= 1$$

5. Time evolution of states.
Before show you the final answer, let us see what is the general rule and how can we derive it.
We denote time evolution as

$$|\psi(t) > \longrightarrow |\psi'(t') >$$

and request time evolution respect superposition, i.e.
if $|\psi(t) > \longrightarrow |\psi'(t') >$ and $|\chi(t) > \longrightarrow |\chi'(t') >$,
then $(\alpha|\psi(t) > +\beta|\chi(t) >) \longrightarrow (\alpha|\psi'(t') > +\beta|\chi'(t') >)$
The consequence is that time evolution is a linear operator on Hilbert space.

6. Probability is conserved by time evolution.
Assume {Finite Time Evolution}=$\sum${infinitesimal evolutions}, let us study the infinitesimal one.

$$|\psi > \xrightarrow[dt]{} |\psi > +iH|\psi > dt$$

$$\Rightarrow < \psi|\psi >= (< \psi| - i < \psi|H dt)\,(|\psi > +iH|\psi > dt)$$

$$=< \psi|\psi > +dt < \psi|H - H^\dagger|\psi > +\mathcal{O}(dt^2)$$

$$\Rightarrow < \psi|H^\dagger|\psi >=< \psi|H|\psi >$$

How far can we go from here? A trick is using superposition principle, since $|\psi >$ is arbitrary, let us write it as $\psi >= \alpha|\phi > +\beta|\chi >$, then the previous result becomes

$$< \phi|H|\chi >=< \phi|H^\dagger|\chi >$$

$$\Rightarrow H = H^\dagger$$

Thus, we can reexpress time evolution using this Hermition operator H aka Hamiltonian as

$$i\partial_t|\psi >= \hat{H}|\psi >$$

7. The last but not least important conception is conserved observable, suppose $\mathcal{O}$ is conserved, this mean its expectation values(or eigenvalues equivalent) respect time evolution. From now on, we can start form here and directly say $\mathcal{O}$ commutes with Hamiltonian, the proof is simple:

$$\text{if } [\mathcal{O}, H] = 0$$

$$\Rightarrow \mathcal{O}H|\psi >= H\mathcal{O}|\psi >$$

$$\Rightarrow \mathcal{O}|\psi' >= H\mathcal{O}|\psi >$$

$$\Rightarrow \mathcal{O}|\psi' >= \lambda H|\psi >= \lambda|\psi' >$$

2 Special Relativity

Now, let us move to special relativity, to make life simple, start with 1 spatial+ 1 temporal dimension case. Then Lorentz transformation becomes:

$$\begin{cases} t' = \dfrac{1}{\sqrt{1-v^2}}(t+vx) \\ \\ x' = \dfrac{1}{\sqrt{1-v^2}}(x+vt) \end{cases}$$

This transformation is followed by requiring of symmetry between two observer(aka related velocity) and the principle that speed of light is the same in each frame. From now on, we will just simply say "c=1".

Go to realistic 3+1 dimensional space time, we will have a bigger symmetry group which, (1) keeps $(x,t) = (0,0)$ origin fixed, and (2) contains rotation and boost.

1. This group is called Lorentz group or simply $SO(1,3)$.

Mathematically, we denoted Lorentz transformation as:

$$x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu}$$

Define rasing and lowering tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we can rase or lower vector as

$$V^{\mu} = \eta^{\mu\nu} V_{\nu}$$

2. The 4 by 4 matrices Λ satisfy

$$\Lambda^T \eta \Lambda = \eta, \qquad \text{and} \qquad \det\Lambda = 1.$$

or explicitly

$$\Lambda^{\mu}{}_{\nu} \eta_{\mu\alpha} \Lambda^{\alpha}{}_{\beta} = \eta_{\nu\beta}$$

3. SO(1,3) contains rotation,

$$\Lambda^T I \Lambda = I \quad \Rightarrow \quad \Lambda^T \Lambda = I$$

This kind of transformation fix length $\sum\limits_i x_i x_i$.
While full Lorentz transformation fix "length" $x^{\mu} x_{\mu}$.

4. Group property of SO(1,3)

$$\begin{cases} I\Lambda = \Lambda I = \Lambda \\ \Lambda_1(\Lambda_2\Lambda_3) = (\Lambda_1\Lambda_2)\Lambda_3 \\ \Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = I \end{cases}$$

5. The full group of relativity transformation is Poincaré Group, generated by space time translations and Lorentz transformation, under which

$$x'^{\mu} = \Lambda^{\mu} x^{\nu} + a^{\mu}$$

6. Meaning of relativity

Laws in x'^{μ} coordinates has identical form to those in x^{μ} coordinates. i.e, `No Prefered Frame`.

NOTES: pay attention, laws here only means EOM, not solution upon BC.

3 What is Relativistic Quantum Mechanics

1. Poincaré transformation acts on Physical Hilbert States.

2. We insist that they respect superpositions

$$G[\alpha|\psi> + \beta|\chi>] = \alpha G|\psi> + \beta|\chi>$$

i.e. G is a linear operator on Hilbert space.

3. Probability Conservation.

$$\text{Under} \quad \alpha|\psi> + \beta|\chi> \to \alpha G|\psi> + \beta G|\chi>$$

Total Probability is

$$(\alpha^* <\psi| + \beta^* <\chi|)\,(\alpha|\psi> + \beta|\chi>) \to \left(\alpha^* <\psi|G^\dagger + \beta^* <\chi|G^\dagger\right)(\alpha G|\psi> + \beta G|\chi>)$$

We demand transformation to conserve total probability(i.e. same for different frames)

$$\Rightarrow <\psi|G^\dagger G|\psi> = <\psi|\psi>$$

Again, using superposition trick:

$$\Rightarrow <\psi|G^\dagger|\chi> + <\chi|G^\dagger G|\chi> = <\psi|\chi> + <\psi|\chi>$$

Now, we make a choice (another one is not "wrong", check Weinberg or Wigner's paper if you want, in fact, that has something to do with time reflection):

$$<\chi|G^\dagger G|\psi> = <\chi|\psi>$$

$$\Rightarrow G^\dagger G = I$$

Thus we know such transformation is unitary.

Example. Time evolution is unitary, and it is inside Poincaré group.

4. Point Particle Theory

I am a particle theorist, so I want to talk about field theory of particle in this course. How to embed concept of point-like particle into relativistic quantum theory we just discussed? You will find our some physics aspect become non-trivial. First, let us write down a N-point quantum mechanics wave function, say it is describing a system containing N particles.

$$\psi(\vec{x_1}, ...\vec{x_n}, t)$$

At first sight, coordinates x and t are not sharing same footing, we admit it, and claim this wave function only works for some specific frame we have chosen. From the point of view inside this frame, at fixed time t, we have a well defined quantum mechanics, e.g. the total probability is 1:

$$\sum |\psi|^2 = 1$$

This condition is true for any fixed time, thus we can slice full space time and wave-function distribution on it(call it "mountain")into simultaneous slices, either contain same mountain volume.

Now, ask what looks like when we go to a different reference frame, Lorentz boost will quench spatial and temporal axis, but we still have probability conservation in this new frame:

$$\sum |\psi'|^2 = 1$$

This is non-trivial from geometric point of view (how can we guarantee "mountain" can be always sliced equalvolumely). In fact more work about theory structure has to be done in order to satisfy this condition.

4 Propagator or Green's Function

1. Propagator in Reality

Let us consider a thought experiment, a "gun" is firing particles. The gun to us is like a black box, we don't want to open it. So how can we know particles are produced there? That is why we need a detector, located away from the gun. Detector receives particles, so we know gun "was" firing.

Now, if we know at time $t = 0$, a particle is produced at location $\vec{x} = 0$, let us ask where is it later.

Since particle was produced locally, we consider a delta wave-function:

$$\psi(x, t=0) = \delta^3(\vec{x})$$

The normalization is well known

$$\int |\psi|^2 dx^3 = \delta^3(0) = V$$

At a later time, this wave-function will time evolute as

$$\begin{aligned} \psi(\vec{x}, t>0) &= e^{i\hat{H}t}\delta^3(\vec{x}) \\ &= \int đp^3 e^{iHt} e^{i\vec{p}\cdot\vec{x}} \\ &= \int đp^3 e^{-i\frac{\vec{p}\,^2}{2m}t} e^{i\vec{p}\cdot\vec{x}} \end{aligned}$$

We can generalize this result into full time region, using named theta-function

$$\psi(\vec{x}, t) = \int đp^3 e^{-i\frac{\vec{p}\,^2}{2m}t} e^{i\vec{p}\cdot\vec{x}}\theta(t)$$

This result is also called Green's Function, $G(x^\mu)$

We are more often interested in Green's Function's Fourier transformation, in order to get that, a integration representation of theta-function is useful:

$$\boxed{\theta(t) = \lim_{\epsilon\to 0^+} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + i\epsilon} e^{-i\omega t}}$$

This integration is evaluate on the complex plane as following: if $t > 0$, we complete the contour below the real axis, while $t < 0$, we complete the contour in the upper half plane.

Translation invariance provides $G(x^\mu) = G(x_2^\mu - x_1^\mu)$, now we can perform Fourier transformation

$$G(x) = \int đp^4 \frac{ie^{-ip\cdot x}}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon} = \int đp^4 G(p) e^{-ip\cdot x}$$

this can be easy understood if we think

$$\omega = p_0 - \frac{\vec{p}\,^2}{2m}$$

The Fourier transformation or momentum representation of Green's function is

$$G(p) = \frac{i}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon}$$

Why this function is important? Because Green's function can help us solve inhomogeneous differential equation. And in this case, help us to get time evolution of initial wave-function

$$\text{if} \quad \psi(\vec{x},0) = \phi(\vec{x}) = \int d\vec{x'}^3 \delta^3(\vec{x}-\vec{x'})\phi(\vec{x'})$$

$$\text{then} \quad \psi(\vec{x},t) = \int d\vec{x'}^3 G(\vec{x}-\vec{x}',t)\phi(\vec{x'})$$

This can be understood as superposition of time evolution, where ϕ takes role of spectrum while δ is the mode evolving.

source with varing amplitude $-iJ_G(\vec{x'},t')$ to provide particle at $(\vec{x'},t')$ gives

$$\psi(\vec{x},t) = -i\int dx^4 G(x-x')J_G(x')$$

$$\text{Ampl}_{G\to D} = (-i)^2 \int dx'^4 \int dx^4 J_D(x)G(x-x')J_G$$

e.g. $\psi(\vec{x},t=0)$ doesn't have to be $\delta^3(\vec{x})$, it could be distribution like Gaussian wave package:

$$e^{-(\vec{x}-\vec{x_0})^2/2a}$$

Fourier transformation of this distribution is also Gaussian type, $e^{-(p-p_0)^2}$ which is the feature of this wave package.

2. What is the difference between gun and detector?

A quick answer is they are exchanged by time reflection $t \to -t$, however there is a loop hole here until we talk about relativistic theory. So far, we can say:

Gun gives particle energy $E > 0$

Detector absorbs particle energy $E < 0$

NOTES: This also contains loop hole, we don't have idea what "energy" is. The assumption here is, particle is "on-shell" near gun and detector.

3. Green's function of inhomogeneous Schödinger equation

Using Green's function method, Shoödinger equation

$$\left(i\partial_t - \hat{H}\right)\psi(x) = -iJ_G(x)$$

can be solved as

$$\psi(\vec{x},t) = \int dx'^4 G(x-x')\Big(-iJ_G(x')\Big)$$

where Green's function

$$G(x) = i \int đp^4 \frac{e^{-ip\cdot x}}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon}$$

proof:

First of all, $$\Big(i\partial_t - \hat{H}\Big)G(x) = i \int đp^4 \frac{(p_0 - \frac{\vec{p}^{\,2}}{2m})e^{-ip\cdot x}}{p_0 - \frac{\vec{p}^{\,2}}{2m} + i\epsilon} = -i\delta^4(x)$$

then $$\begin{aligned}\Big(i\partial_t - \hat{H}\Big)\psi(x) &= \int dx'^4 \Big((i\partial_t - \hat{H})G(x-x')\Big)\Big(-iJ_G(x')\Big) \\ &= \int dx'^4 \delta(x-x')\Big(-iJ_G(x')\Big) = -iJ_G(x)\end{aligned}$$

□

A point I want to make here is that this inhomogeneous Schödinger equation in fact does not preserve probability. The main reason is that we treat the gun(a.k.a. source) as a black box, We can not require probability(or any global quantity like energy) to be conserved, unless we also included everything inside the gun into our system. Talking about conservation in a open system usually is unreasonable.

We will tempting to call p_0 the energy of the particle. However, the catch here is generally $p_0 \neq \frac{\vec{p}^2}{2m}$. This aspect is called "off mass shell", while for those $p_0 = \frac{\vec{p}^2}{2m}$ cases, we say particle is on-shell.

You may wonder what is going on here. The deeper answer is Uncertainty Principle, a.k.a. $\Delta E \cdot \Delta t \geq 1$, which warns us that if $t_D - t_G$ is small, we will have large ΔE as a consequence.

Let us rewrite the time evolving wave function

$$\psi(\vec{x}, t) = \int dx'^4 \int đp^4 \frac{e^{-ip\cdot x}}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon} J_G(x')$$

Consider its asymptotic behavior, i.e. when x^μ is large, $e^{ip\cdot x}$ is a wild phase, the total contribution then is 0 when rest integrand is smoothly varing.

However, there is still small region can give a finite contribution even in this asymptotic limit, roughly speaking, when the denominator of integrand close to zero, i.e. $p_0 \sim \frac{\vec{p}^2}{2m}$, integrand also rapidly changing so that wild phase cancelation is evaded.

Mathematically, this has something to do with $\theta(t)$, this jump function provide a definite value which can be evaluated out after intergration.

If we try to make physics sense out of expression, we must look inside the gun, the previous black box. We find there IS an uncertainty there already, e.g. how can a particle start to appear and how can it start to move. Without touching philosophy, we still can write down a theory effectively correct, and include this "initial condition", a simplest way to do is write source wave function as:

$$\theta(-t)\exp\frac{-i\vec{q}^{\,2}}{2m}t+\theta(t)\exp\frac{-i\vec{p}^{\,2}}{2m}t \quad \text{while p is large and q is small}$$

This expression is nothing but a fancy way to tell the following story:
A particle is almost localized near origin, then after a sudden kick, is moving in a higher speed. Now you see change of motion is always related with a step function. We have learned impulse in classical mechanics, there we call the similar principle "Newton's Law".

4. An analysis in mixed position-momentum space is helpful here. In this representation, only one of space-time variable has been Fourier transformed.

$$G(t,\vec{p}) = ie^{-i\frac{\vec{p}^2}{2m}\cdot t}\int dE\frac{e^{-i(E-\frac{\vec{p}^2}{2m})t}}{E-\frac{\vec{p}^{\,2}}{2m}+i\epsilon}$$

Now, the condition for wild phase becomes:

$$(E-\frac{\vec{p}^{\,2}}{2m})t \gg 1$$

$$\text{or}\quad E-\frac{\vec{p}^{\,2}}{2m} \gg \frac{1}{t}$$

Let me summrize what we got here:

Particle is generically off-shell while propagating, which is a consequence of uncertainty principle, we can measure its on-shell value if and if we wait enough long time, or put the detetor far away. Later this will be called *external line* in this course.

We don't really know what changes particle's motion and how it really works, however, using effective description involved θ function, we can simply tell particle's momentum has been changed from intial value to some final value, this later will be called interaction *vertex.* lines joint at vertex carry difinite momentum value, they are not necessarily on-shell unless they are external ones.

External lines, propagators(internal lines), and vertices are building blocks of quantum field theory description.